Final Exam 2012 Winter Term 2 Solutions

1. (a) Find an equation of the plane which is parallel to the plane $Q : -2x + y = 3z + 1$ and passes through the point $(-1, 1, 2)$.

Solution: For $Q : -2x+y-3z = 1$, we get the normal vector $\mathbf{n}_Q = \langle -2, 1, -3\rangle$. Since the plane is parallel to Q, we may use the same normal vector. Thus, an equation of the plane which is parallel to the plane $Q : -2x + y = 3z + 1$ and passes through the point $(-1, 1, 2)$ is:

$$-2(x + 1) + (y - 1) + -3(z - 2) = 0 \Leftrightarrow -2x + y - 3z = -3.$$

(b) Find an equation for the level curve of $f(x, y) = 3xy^2 + 2y - 1$ that goes through the point $(1, -2)$.

Solution: Since $f(1, -2) = 3(1)(-2)^2 + 2(-2) - 1 = 7$, an equation for the level curve of $f(x, y) = 3xy^2 + 2y - 1$ that goes through the point $(1, -2)$ is:

$$7 = 3xy^2 + 2y - 1.$$

(c) Evaluate $\displaystyle\int_0^{5/2} \frac{dx}{\sqrt{25 - x^2}}$.

Solution: Using trigonometric substitution with $x = 5\sin\theta$, $dx = 5\cos\theta d\theta$, when $x = 0$, $\theta = 0$, when $x = \frac{5}{2}$, then $\sin\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$, we get:

$$\int_0^{5/2} \frac{dx}{\sqrt{25 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{5\cos\theta d\theta}{5\cos\theta} = \int_0^{\frac{\pi}{6}} d\theta = \theta\ |_0^{\frac{\pi}{6}} = \frac{\pi}{6}.$$

(d) Fill in the blanks with right,left, or midpoint; an interval, and a value of n: $\sum_{k=0}^{3} f(1.5 + k) \cdot 1$ is a ? Riemann sum for f on the interval [?, ?] with n =?.

Solution: Note that the usual form of Riemann sum is $\sum_{k=1}^{n} f(x_k^*)\Delta x$ which starts at $k = 1$, so we first want to change the index:

$$\sum_{k=0}^{3} f(1.5 + k) \cdot 1 = \sum_{k=1}^{4} f(0.5 + k) \cdot 1.$$

So, $n = 4$, $\Delta x = 1$ and $x_k^* = 0.5 + k$. We may choose sum to be a right Riemann sum, so $x_k^* = x_k = a + k\Delta x$. So, $a = 0.5$ and $b = a + n\Delta x = 2.5$. Thus, $\sum_{k=0}^{3} f(1.5 + k) \cdot 1$ is a right Riemann sum for f on the interval $[0.5, 2.5]$ with $n = 4$.

<u>Remark:</u> *The only fixed solution to this is* $n = 4$. *The answers to other parts are not unique. We can also say that the sum is a left Riemann sum on* $[1.5, 3.5]$ *or a midpoint Riemann sum on* $[1, 3]$.

(e) Evaluate $\int_{-1}^{2} |2x|\,dx$.

Solution: Since

$$|2x| = \begin{cases} -2x & \text{for } x \le 0, \\ 2x & \text{for } x > 0, \end{cases}$$

we get:

$$\int_{-1}^{2} |2x|\,dx = \int_{-1}^{0} -2x\,dx + \int_{0}^{2} 2x\,dx = -x^2 \,\big|_{-1}^{0} + x^2 \,\big|_{0}^{2} = 1 + 4 = 5.$$

(f) If $F(x) = \int_{0}^{\cos x} \frac{1}{t^3 + 6}\,dt$, find $F'(\pi)$.

Solution: Using Fundamental Theorem of Calculus Part I and chain rule, we get:

$$F'(x) = \frac{1}{\cos^3(x) + 6}(-\sin(x)),$$
$$F'(\pi) = \frac{1}{\cos^3(\pi) + 6}(-\sin(\pi)) = 0.$$

(g) Find the area of the region bounded by the graph of $f(x) = \frac{1}{(2x-4)^2}$ and the x-axis between $x = 0$ and $x = 1$.

Solution: Note that this area entirely lies above the x-axis, so it is the same as the net area. Hence, to find the area, we may compute the following integral after a direct substitution with $u = 2x - 4$, $du = 2dx$:

$$\int_0^1 \frac{dx}{(2x-4)^2} = \int_{-4}^{-2} \frac{1}{2} u^{-2}\, du = -\frac{1}{2u} \Big|_{-4}^{-2} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

(h) Evaluate $\displaystyle\int \frac{\ln(x)}{x^7}\, dx$.

Solution: Using integration by parts with $u = \ln x$, $du = \frac{dx}{x}$, and $dv = \frac{dx}{x^7}$, $v = -\frac{1}{6x^6}$, we get:

$$\int \frac{\ln(x)}{x^7}\, dx = -\frac{\ln x}{6x^6} + \int \frac{dx}{6x^7} = -\frac{\ln x}{6x^6} - \frac{1}{36x^6} + C.$$

(i) Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{e^k k!}$.

Solution: Note that:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \text{ for all } x.$$

So,

$$\sum_{k=0}^{\infty} \frac{1}{e^k k!} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{e}\right)^k}{k!} = e^{\left(\frac{1}{e}\right)}.$$

(j) Find a bound for the error in approximating $\displaystyle\int_1^5 \frac{1}{x}\, dx$ using Simpson's rule with $n = 4$. Do not write down Simpson's rule approximation S_4.

Solution: For $a = 1$,$b = 5$, $n = 4$, we get $\Delta x = \frac{b-a}{n} = 1$. Finding the derivatives of $f(x)$:

$$f'(x) = -x^{-2}, \quad f''(x) = 2x^{-3}, \quad f'''(x) = -6x^{-4}, f^{(4)}(x) = 24x^{-5}.$$

So, $|f^{(4)}(x)| = 24(|x|)^{-5}$. To find K, note that $1 \le x \le 5$, so $1 \le |x| \le 5$, and $1 \le |x|^5 \le 5^5$. Taking the reciprocal, we get: $1 \ge |x|^{-5} \ge 5^{-5}$, so $24 \ge 24|x|^{-5} \ge 24(5^{-5})$. Hence, for $K = 24$, we have that $|f^{(4)}(x)| \le K$ for all x in $[1, 5]$. Thus, an error bound is:

$$E_4 \le \frac{K(b-a)(\Delta x)^n}{180} = \frac{24(4)1^4}{180} = \frac{8}{15}.$$

(k) Find the Maclaurin series for $f(x) = \dfrac{1}{2x-1}$.

Solution: We have that:

$$\frac{1}{2x-1} = -\frac{1}{1-2x} = -\sum_{k=0}^{\infty}(2x)^k = \sum_{k=0}^{\infty} -2^k x^k,$$

for $|2x| < 1$.

(l) Let k be a constant. Find the value of k such that $f(x) = ke^{-x}e^{(-e^{-x})}$ is a probability density function on $(-\infty, 1]$.

Solution: In order to be a probability density function, we need $\int_{-\infty}^{\infty} f(x)\,dx = 1$, so:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} ke^{-x}e^{(-e^{-x})}\,dx \quad (\text{ since } f(x) = 0 \text{ outside of } (-\infty, 1])$$
$$= \lim_{a\to-\infty}\int_{a}^{1} ke^{-x}e^{(-e^{-x})}\,dx.$$

Using $u = -e^{-x}$, $du = e^{-x}dx$, we get:

$$1 = \lim_{a\to-\infty}\int_{-e^{-a}}^{-e^{-1}} ke^{u}\,du = \lim_{a\to-\infty} ke^{u}\,\Big|_{-e^{-a}}^{-e^{-1}}$$
$$\Rightarrow 1 = \lim_{a\to-\infty} ke^{(-e^{-1})} - ke^{(-e^{-a})} = ke^{(-e^{-1})}.$$

So, $k = e^{(e^{-1})}$.

(m) Compute the cumulative distribution function corresponding to the probability density function $f(x) = 3x^{-4}$, $x \geq 1$.

Solution: Let $F(x) = \int_{-\infty}^{x} f(t)\,dt$ be the cumulative distribution function of $f(x)$. Since $f(x) = 0$ for $x < 1$, we get $F(x) = 0$ for $x < 1$. For $x \geq 1$,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{1}^{x} 3t^{-4}\,dt = -t^{-3}\,\Big|_{1}^{x} = -x^{-3} + 1.$$

Thus, the cumulative distribution function corresponding to $f(x)$ is:

$$F(x) = \begin{cases} 0, & x < 1, \\ -x^{-3} + 1, & x \geq 1. \end{cases}$$

(n) Find the expected value of the random variable X whose probability density function is $f(x) = \frac{2}{9\sqrt[3]{x}}, 1 \leq x \leq 4$.

Solution: We have that:

$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_1^4 x\frac{2}{9\sqrt[3]{x}}dx = \int_1^4 \frac{2}{9}x^{2/3}\,dx \\ &= \frac{2}{15}x^{5/3}\,|_1^4 = \frac{2}{15}\sqrt[3]{4^5} - \frac{2}{15}.\end{aligned}$$

2. (a) $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2 - k}$.

Solution: We want to use the Limit Comparison Test with the second series being $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2}$. We have that:

$$\lim_{k\to\infty} \frac{a_n}{b_n} = \lim_{k\to\infty} \frac{\sqrt[3]{k}}{k^2 - k} \cdot \frac{k^2}{\sqrt[3]{k}} = \lim_{k\to\infty} \frac{1}{1 - \frac{1}{k}} = 1.$$

So, either both series converge or both diverge. Note that $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{5}{3}}}$, which is a p-series with $p = \frac{5}{3} > 1$. Hence, this series converges by the p-series Test, and therefore, $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2 - k}$ also converges.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{10}10^k(k!)^2}{(2k)!}$.

Solution: We want to use the Ratio Test:

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{n\to\infty} \frac{(k+1)^{10}10^{k+1}((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{k^{10}10^k(k!)^2} \\ &= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{10} \cdot \frac{10^{k+1}}{10^k} \cdot \frac{((k+1)!)^2}{(k!)^2} \cdot \frac{(2k)!}{(2(k+1))!} \\ &= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{10} \cdot 10 \cdot (k+1)^2 \cdot \frac{1}{(2k+1)(2k+2)} \\ &= \frac{10}{4} > 1.\end{aligned}$$

Thus, the series diverges.

(c) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)(\ln \ln k)}$.

Solution: We want to use Integral Test with $f(x) = \dfrac{1}{x(\ln x)(\ln \ln x)}$. To compute the improper integral, we need to use a direct substitution with $u = \ln \ln x$, and $du = \dfrac{1}{x(\ln x)} dx$. We get:

$$\int_3^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x(\ln x)(\ln \ln x)} dx = \lim_{b\to\infty} \int_{\ln\ln 3}^{\ln \ln b} \frac{1}{u}\,du$$
$$= \lim_{b\to\infty} \ln |u| \Big|_{\ln\ln 3}^{\ln\ln b} = \lim_{b\to\infty} \ln|\ln\ln b| - \ln|\ln\ln 3| = \infty.$$

So, since the improper integral diverges, the series $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)(\ln \ln k)}$ also diverges by the Integral Test.

3. (a) Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{1}{(x^2+x)y}, \qquad y(1) = 2.$$

Solution: Separating the variables and then integrating each side with respect to the corresponding variables, we get:

$$\frac{dy}{dx} = \frac{1}{(x^2+x)y} \Leftrightarrow ydy = \frac{dx}{x^2+x} \Rightarrow \int ydy = \int \frac{dx}{x^2+x}.$$

We have that $\displaystyle\int ydy = \frac{y^2}{2} + C$. For $\displaystyle\int \frac{dx}{x^2+x}$, note that $x^2 + x = x(x+1)$, so we can use partial fractions. Set:

$$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{(A+B)x + A}{x(x+1)}.$$

So, $A = 1$ and $A + B = 0$ which yields $B = -1$. Hence,

$$\int \frac{dx}{x^2+x} = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \ln|x| - \ln|x+1| + C.$$

Since $y(1) = 2$, we get:

$$\frac{2^2}{2} = \ln|1| - \ln|1+1| + C \Rightarrow 2 = C - \ln 2 \Rightarrow C = 2 + \ln(2).$$

Hence, the solution to the initial value problem is:

$$\frac{y^2}{2} = \ln|x| - \ln|x+1| + 2 + \ln(2).$$

(b) Let $f(0) = 1, f(2) = 3$, and $f'(2) = 4$. Calculate $\displaystyle\int_0^4 f''(\sqrt{x})\,dx$.

Solution: First, using a direct substitution with $u = \sqrt{x}$, and $du = \frac{dx}{2\sqrt{x}}$, that is, $2udu = dx$, we get:

$$\int_0^4 f''(\sqrt{x})\,dx = \int_0^2 uf''(u)\,du.$$

Using integration by parts with $u_1 = u$, $du_1 = du$, and $dv_1 = f''(u)\,du$, $v_1 = f'(u)$, we get:

$$\begin{aligned}\int_0^2 uf''(u)\,du &= uf'(u)\ |_0^2 - \int_0^2 f'(u)\,du = 2f'(2) - f(u)\ |_0^2 \\ &= 2f'(2) - f(2) + f(0) = 2(4) - 3 + 1 = 6.\end{aligned}$$

Hence, $\displaystyle\int_0^4 f''(\sqrt{x})\,dx = 6$.

4. Let $f(x,y) = xye^{-2x-y}$.

(a) Find all critical points of $f(x,y)$.

Solution: Note that $f(x,y)$ is defined everywhere on $\mathbb{R}^2$, so the only critical points are those where both partial derivatives are zero.

$$\begin{aligned}f_x(x,y) &= ye^{-2x-y} - 2xye^{-2x-y} = (1-2x)ye^{-2x-y},\\ f_y(x,y) &= xe^{-2x-y} - xye^{-2x-y} = (1-y)xe^{-2x-y}.\end{aligned}$$

$f_x(x,y) = 0$ implies that $x = 1/2$ or $y = 0$, and $f_y(x,y) = 0$ implies that $y = 1$ or $x = 0$. Thus, we get two critical points $(1/2, 1)$ and $(0,0)$.

(b) Classify each critical point you found as a local maximum, a local minimum, or a saddle point of $f(x,y)$.

Solution: We want to find all second order partial derivatives of f:

$$f_{xx}(x,y) = -2ye^{-2x-y} - 2(1-2x)ye^{-2x-y} = 4(x-1)ye^{-2x-y},$$
$$f_{yy}(x,y) = -xe^{-2x-y} - (1-y)xe^{-2x-y} = (y-2)xe^{-2x-y},$$
$$f_{xy}(x,y) = (1-2x)e^{-2x-y} - (1-2x)ye^{-2x-y} = (1-2x)(1-y)e^{-2x-y}.$$

Note that the discriminant $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$. Using the Second Derivative Test, we get:

- At $(1/2, 1)$, $f_{xx}(1/2,1) = -2e^{-2} < 0$, $f_{yy}(1/2,1) = -1/2e^{-2}$, $f_{xy}(1/2,1) = 0$, and $D(1/2,1) = e^{-4} > 0$, we get that $(1/2,1)$ is a local maximum.
- At $(0,0)$, $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 1$, and $D(0,0) = -1 < 0$. So, $(0,0)$ is a saddle point.

5. (a) A study conducted at a waste disposal site reveals soil contamination over a region that can be described as the interior of the circle $x^2 + y^2 = 16$, where x and y are in miles. In order to build a circular enclosure to contain all polluted territory, the manager of the site wants to find the radius of the smallest circle centered at $(2,2)$ that contains the entire contamination region. Formulate this as a constrained optimization problem, clearly stating the objective function and the constraint. Note that you do not need to do any computation in part (a).

Solution: In order to minimize the radius of the circle centered at $(2,2)$ that contains the contamination region, we want to have that circle to intersect with the circle $x^2+y^2 = 16$ at exactly one point, in which case, the radius is precisely the distance between that point and $(2,2)$, given by $\sqrt{(x-2)^2 + (y-2)^2}$. Furthermore, to ensure that the entire contamination region lies within the circle centered at $(2,2)$, that distance should be maximal among all points that lies on the circle $x^2 + y^2 = 16$. Thus, the problem can be re-formulated as follows: we want to find the maximum value of $f(x,y) = \sqrt{(x-2)^2 + (y-2)^2}$ subject to the constraint $g(x,y) = x^2 + y^2 - 16 = 0$.

(b) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 6y - y^3 - 3x^2y$ on the circle $x^2 + y^2 = 4$.

Solution: By Lagrange multipliers, we want to solve the following system of equations:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} -6xy = \lambda 2x \\ 6 - 3y^2 - 3x^2 = \lambda 2y \\ x^2 + y^2 - 4 = 0 \end{cases}$$

From the first equation, we get $2x(1+3y) = 0$, so $x = 0$ or $y = -\frac{1}{3}$. Consider two cases:

- If $x = 0$, then the third equation gives $y = \pm 2$. If $y = 2$, then $6 - 18 = 4\lambda$, so $\lambda = -3$. If $y = -2$, then $6 - 18 = -4\lambda$, and $\lambda = 3$.
- If $y = -\frac{1}{3}$, then the third equation gives $x = \pm\frac{\sqrt{35}}{3}$. So,

$$6 - \frac{1}{3} - 35 = -\frac{2}{3}\lambda \Rightarrow \lambda = 44.$$

Computing the value of $f(x,y)$ at those points:

$$f(0,2) = 4, \quad f(0,-2) = -4, \quad f\left(\frac{\sqrt{35}}{3}, -\frac{1}{3}\right) = \frac{52}{27}, \quad f\left(-\frac{\sqrt{35}}{3}, -\frac{1}{3}\right) = \frac{52}{27}.$$

So, on the circle $x^2 + y^2 = 4$, the maximum value of $f(x,y)$ is 4 and the minimum value of $f(x,y)$ is -4.

6. Find the interval of convergence of the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^{2k}}{k^2 9^k}$.

Solution: By the Ratio Test,

$$L = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x+1)^{2(k+1)}}{(k+1)^2 9^{k+1}} \cdot \frac{k^2 9^k}{(x+1)^{2k}}\right|$$
$$= \lim_{k\to\infty}\frac{k^2|x+1|^2}{9(k+1)^2} = \frac{|x+1|^2}{9}.$$

The series converges for $L < 1$, that is, $\frac{|x+1|^2}{9} < 1$, or equivalently, $|x+1| < 3$. Hence, the radius of convergence is 3. Now, we need to test the convergence at the two endpoints $x = 2$ and $x = -4$.

- At $x = 2$, the series becomes: $\displaystyle\sum_{k=1}^{\infty}\frac{3^{2k}}{k^2 9^k} = \sum_{k=1}^{\infty}\frac{1}{k^2}$, which converges by a

p-series Test with $p = 2 > 1$.

- At $x = -4$, the series becomes: $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^{2k}}{k^2 9^k} = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which also converges by a p-series Test.

Thus the interval of convergence is $[-4, 2]$.

(b) $\displaystyle\sum_{k=1}^{\infty} a_k (x-1)^k$, where $a_k > 0$ for $k = 1, 2, 3, \ldots$, and

$$\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \frac{a_1}{a_2}.$$

Solution: By the Ratio Test, we want to find:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}(x-1)^{k+1}}{a_k (x-1)^k} \right| = |x-1| \lim_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

Since $\displaystyle\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \frac{a_1}{a_2}$, by definition, $\displaystyle\lim_{n\to\infty} s_n = \frac{a_1}{a_2}$, where s_n is the n-th partial sum defined by:

$$s_n = \sum_{k=1}^{n} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}} \right) = \left(\frac{a_1}{a_2} - \frac{a_2}{a_3} \right) + \ldots + \left(\frac{a_n}{a_{n+1}} - \frac{a_{n+1}}{a_{n+2}} \right) = \frac{a_1}{a_2} - \frac{a_{n+1}}{a_{n+2}}.$$

So,

$$\begin{aligned} \frac{a_1}{a_2} &= \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{a_1}{a_2} - \frac{a_{n+1}}{a_{n+2}} \right) = \frac{a_1}{a_2} - \lim_{n\to\infty} \frac{a_{n+1}}{a_{n+2}} \\ \Rightarrow 0 &= \lim_{n\to\infty} \frac{a_{n+1}}{a_{n+2}} \Rightarrow \infty = \lim_{n\to\infty} \frac{a_{n+2}}{a_{n+1}}. \end{aligned}$$

Hence,

$$L = |x-1| \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty.$$

Since the series converges if $L < 1$, we get that the radius of convergence is 0. Note that at $x = 1$, the series is identically 0, so the interval of convergence is just one point $x = 1$.